The numerical series $\sum_{1,\infty}$ converges under the condition $0 < t < 1$ assumed, consequently, the sequence of its partial sums is fundamental. Therefore, the last expression tends to zero as $m \to \infty, n \to \infty$ in the chain of estimates presented.

Summarizing, the solution of the thermoelasticity problem for an inhomogeneous body can be sought by the perturbation method in the form of the series (2.2) that converges in the energy space metric and in theequivalent metric $L_2(V)$. The stresses $\tau^{(k)}$ are here defined uniquely from the recursion sequence of problems (2.3),(2.4),(2.5).

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EQUILIBRIUM OF A PRESTRESSED ELASTIC BODY WEAKENED BY A PLANE ELLIPTICAL CRACK*

V.M. ALEKSANDROV and B.V. SOBOL'

The problem of normal pressure loading of the edges of a plane elliptical crack is considered. The crack subjected to the load is in the open state. The medium in which it is located is frist subjected to homogeneous biaxial tension or compression along the plane of the crack. A model of incompressible neo-Hooke material is considered /1/. The problem is reduced to solving a singular integral equation of the first kind. In the case when the intensity of the initial loading is identical in both directions, the problem has an exact solution. If the coefficients of preliminary tension differ slightly, construction of the solution of the problem is possible by an asymptotic method /2/. It is shown that as in the case of equal coefficients /3/**, the initial stress does not alter the order of the singularity of the stress field near the crack edge and only affects the normal stress intensity factor. (**See also: Filippova, L.M. On the opening of a circular crack in a prestressed elastic body. Second All-Union Scientific Conference, "Mixed Problems of the Mechanics of a Deformable Body". Abstracts of Reports /in Russian/, Dnepropetrovsk State Univ.., 1981)

Analogous problems are considered in /4, 5/ for the case of equal prestrain coefficients in a body containing a circular crack. A solution /4/ is constructed for the axisymmetric problem for a layer under different conditions on its faces, and it is shown /5/ that it is possible to use the solution of the problem concerning a crack in an anisotropic material. A solution of the axisymmetric problem is constructed /6/ in the case of radial finite prestrain. An asymptotic solution /7/ is obtained for the spatial contact problem for a prestressed elastic body.

1. Let a crack occupying the domain Ω_1 in planform be located in the plane $z = 0$ of an elastic space. Uniform loads $\sigma_x = t_1$ and $\sigma_y = t_2$ act in two mutually perpendicular directions

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*Prikl. Matem. Mekhan.,49,2,348-352,1985 at infinity, causing the finite deformation of a neo-Hooke body (Fig.1). The following equilibrium equations result from the relationships of the theory of small deformations imposed on a finite deformation /1/, and describe the deformation of a prestressed body in the case of a neo-Hooke material /7/:

$$\lambda_1^2 \frac{\partial^2 \mathbf{L}}{\partial x^2} + \lambda_2^2 \frac{\partial^2 \mathbf{L}}{\partial y^2} + \lambda_3^2 \frac{\partial^2 \mathbf{L}}{\partial z^2} + \frac{2}{G} \operatorname{grad} q = 0, \quad \operatorname{div} \mathbf{L} = 0. \tag{1.1}$$

Here $\mathbf{L} = \{u, v, w\}$ is the additional displacement vector, x, y, z are Cartesian coordinates in the prestrained state, q is a function of the additional pressure whose presence is related to the incompressibility of the material, $\lambda_1, \lambda_2, \lambda_3$ are the initial stretch coefficients along the coordinate axes, and G is a constant of the material.

Since the material is incompressible, the relationship $\lambda_1\lambda_2\lambda_3 = 1$ holds /1/ for the principal stretch coefficients. It is assumed that $\sigma_z = 0$ in the prestressed state.

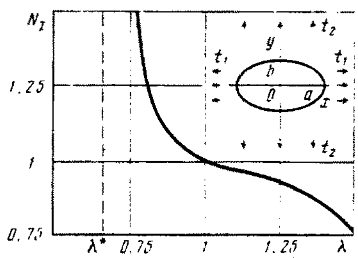

Fig.1

The relationship of the initial stretch coefficients to the stresses has the form

$$t_1 = G(\lambda_1^2 - \lambda_3^2), \quad t_2 = G(\lambda_2^2 - \lambda_3^2).$$

In particular, let the domain occupied by the crack in the unstressed state be elliptical Ω_1: $x^2/a_1^2 + y^2/b_1^2 \leqslant 1$. In this case, as a result of the prestrain the crack occupies the planform domain Ω: $x^2/a^2 + y^2/b^2 \leqslant 1$ whose dimensions are related to the crack initial dimensions as follows: $a = \lambda_1 a_1$ and $b = \lambda_2 b_1$. In particular, the circular crack can take an elliptical form (if $\lambda_1 \neq \lambda_2$) and for an appropriate selection of the prestrain coefficients the elliptical crack can indeed become circular.

We introduce the notation $\sigma_z = -p(x, y)$ for the load applied to the crack edges.

By virtue of the symmetry of the problem about the $z = 0$ plane, we can represent the boundary conditions in this plane in the form

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \tag{1.2}$$

$$q + G\lambda_1^{-2}\lambda_2^{-2}\frac{\partial w}{\partial z} = -\frac{p}{2}: \quad (x, y) \in \Omega, \qquad w = 0; \ (x, y) \notin \Omega.$$

A consequence of (1.1) is the Laplace equation for the function $q(x, y, z)$, which can be used to determine the additional pressure function.

We apply a two-dimensional Fourier integral transform

$$f^*(\alpha, \beta, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z)\, e^{-i(\alpha x + \beta y)}\, dx\, dy \tag{1.3}$$

to (1.1), the Laplace equation $\Delta q = 0$, and the boundary conditions (1.2).

Consequently, we obtain a system of ordinary differential equations with constant coefficients

$$\frac{d^2 u^*}{dz^2} - \lambda_1^2\lambda_2^2\nu^2 u^* = -\frac{2i\alpha\lambda_1^2\lambda_2^2}{G} q^* \tag{1.4}$$

$$\frac{d^2 v^*}{dz^2} - \lambda_1^2\lambda_2^2\nu^2 v^* = -\frac{2i\beta\lambda_1^2\lambda_2^2}{G} q^*$$

$$\frac{d^2 w^*}{dz^2} - \lambda_1^2\lambda_2^2\nu^2 w^* = -\frac{2\lambda_1^2\lambda_2^2}{G}\frac{dq^*}{dz}; \quad \nu^2 = \lambda_1^2\alpha^2 + \lambda_2^2\beta^2$$

$$i(\alpha u^* + \beta v^*) + dw^*/dz = 0 \tag{1.5}$$

$$d^2 q^*/dz^2 - \varkappa^2 q^* = 0, \ \varkappa^2 = \alpha^2 + \beta^2. \tag{1.6}$$

For $z = 0$ the boundary conditions in terms of the Fourier transforms become

$$du^*/dz + i\alpha w^* = 0, \ dv^*/dz + i\beta w^* = 0, \quad q^* + G\lambda_1^{-2}\lambda_2^{-2}\frac{dw^*}{dz} = -\frac{p^*}{2}. \tag{1.7}$$

We obtain directly from (1.6)

$$q^* = Q(\alpha, \beta)\, e^{\varkappa z}. \tag{1.8}$$

This enables us to construct a general solution of the system of differential Eq.(1.4). Solving each separately, we find

$$(u^*, v^*, w^*) = (U(\alpha, \beta), V(\alpha, \beta), W(\alpha, \beta))\, e^{-\lambda_1\lambda_2 \nu z} + (-i\alpha, -i\beta, \lambda_1{}^2\lambda_2{}^2\varkappa)\, Q(\alpha, \beta) \frac{2}{G} \frac{e^{-\varkappa z}}{\varkappa^2 - \lambda_1{}^2\lambda_2{}^2\nu^2} \tag{1.9}$$

Using the incompressibility condition in the form (1.5) we can eliminate $W(\alpha, \beta)$ from (1.9)

$$W(\alpha, \beta) = i \frac{\alpha U(\alpha, \beta) + \beta V(\alpha, \beta)}{\lambda_1\lambda_2\nu}.$$

By satisfying the boundary conditions (1.7), we obtain

$$q^* = \frac{G}{2} N(\alpha, \beta, \lambda_1\lambda_2)\, w^*.$$

Hence, the following integral equation for determining the vertical displacement function for the crack edge results from the passage to Fourier originals:

$$\iint_{\Omega} \gamma(\xi, \eta) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} N(\alpha, \beta, \lambda_1, \lambda_2) \exp\{i[\alpha(x-\xi) - \beta(y-\eta)]\}\, d\alpha\, d\beta\, d\Omega = \frac{4\pi^2}{G} p(x, y) \tag{1.10}$$

$$\gamma(x, y) = w(x, y, 0);\ (x, y) \in \Omega$$

$$N(\alpha, \beta, \lambda_1\lambda_2) = \frac{\varkappa^4 - \lambda_1{}^4\lambda_2{}^4\nu^4 - 2\varkappa^2(\varkappa - \lambda_1\lambda_2\nu)^2}{\varkappa\lambda_1{}^2\lambda_2{}^2(\varkappa^2 - \lambda_1{}^2\lambda_2{}^2\nu^2)}.$$

2. We consider the case when the initial stretch coefficients are identical in both directions, i.e., $\lambda_1 = \lambda_2 = \lambda$. Here evidently $\nu = \lambda\varkappa$ and

$$N(\alpha, \beta, \lambda) = 2T(\lambda)\varkappa;\quad T(\lambda) = \frac{\lambda^9 + \lambda^6 + 3\lambda^3 - 1}{2\lambda^4(\lambda^3 + 1)}.$$

Because of direct calculations /8/, the integral Eq.(1.10) is reduced in the case under consideration to the form

$$T(\lambda)\Delta \iint_{\Omega} \gamma(x, y) \frac{d\Omega}{R} = -\frac{\pi}{R} p(x, y) \tag{2.1}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\quad R = \sqrt{(x-\xi)^2 + (y-\eta)^2}.$$

As is easy to see, the function $T(\lambda)$ increases monotonically for $\lambda^* < \lambda < \infty$; $K(\lambda^*) = 0$, $\lambda^* = 0.667$. It has been established /3, 7/ that as $\lambda \to \lambda^*$ the compressed spaced buckles. This limit case in the problem under consideration corresponds here to fracture of a neo-Hookean body with a crack, irrespective of the magnitude of the applied additional pressure.

If there is no prestrain $(\lambda = 1)$, then the corresponding equation of the classical problem results from integral Eq.(2.1) in the case of an incompressible material (i.e., Poisson's ratio equals $1/2$).

We assume further that the stretch coefficients in the directions of the coordinate axes Ox and Oy are distinct but $\lambda_1 = \lambda + \varepsilon$ and $\lambda_2 = \lambda - \varepsilon$ here, and it is assumed that $\varepsilon/\lambda \ll 1$. In this case the following asymptotic expansion is possible:

$$N(\alpha, \beta, \lambda_1, \lambda_2) = D(\alpha, \beta, \lambda, \varepsilon) = 2T(\lambda)\varkappa - 2M(\lambda)\varepsilon \frac{\alpha^2 - \beta^2}{\varkappa} - O(\varepsilon^2) \tag{2.2}$$

$$M(\lambda) = \frac{\lambda^9 - 2\lambda^6 - \lambda^3 - 2}{\lambda^2(\lambda^3 - 1)^2}.$$

As in the preceding case, such a representation enables the kernel of the integral Eq. (1.10) to be evaluated. Using the results in /8/, we note that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\alpha^2 - \beta^2}{\varkappa} \exp\{i[\alpha(x-\xi) - \beta(y-\eta)]\}\, d\alpha\, d\beta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\pi}{R}.$$

We therefore obtain the following integral equation in the function $\gamma(x, y)$ in the case under consideration

$$T(\lambda)\Delta \iint_{\Omega} \gamma(\xi, \eta) \frac{d\Omega}{R} - \frac{M(\lambda)}{2}\varepsilon \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) \iint_{\Omega} \gamma(\xi, \eta) \frac{d\Omega}{R} + \tag{2.3}$$

$$O(\varepsilon^2) = -\frac{\pi}{G} p(x, y);\quad (x, y) \in \Omega.$$

The integral Eq.(2.2) is obtained from (2.3) by passing to the limit as $\varepsilon \to 0$.

3. We will now construct the asymptotic solution of (2.3) in the case of an elliptical domain Ω. We here seek the vertical displacement function for the crack edges in the form

$$\gamma(x, y) = \sum_{n=0}^{\infty} \gamma_n(x, y)\,\varepsilon^n \tag{3.1}$$

Substituting (3.1) into (2.3) and equating expressions for identical powers of ε, we obtain a system of integral equations to determine the first two terms in expansion (3.1):

$$T(\lambda)\,\Delta \iint_{\Omega} \gamma_0(\xi, \eta)\frac{d\Omega}{R} = -\frac{\pi}{G}\,p(x, y); \qquad (x, y) \in \Omega \tag{3.2}$$

$$T(\lambda)\,\Delta \iint_{\Omega} \gamma_1(\xi, \eta)\frac{d\Omega}{R} = \frac{M(\lambda)}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\iint_{\Omega} \gamma_0(\xi, \eta)\frac{d\Omega}{R}.$$

Using the result obtained in /9/ we can show that for

$$p(x, y) = \sum_{i=0}^{r}\sum_{j=0}^{l} p_{ij}x^i y^j \begin{pmatrix} r + l = n \\ p_{ij} - \mathrm{const} \end{pmatrix}$$

the solution of the first equation of (3.2) for the elliptical domain Ω has the form

$$\gamma_0(x, y) = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} \sum_{i=0}^{s}\sum_{j=0}^{m} g_{ij}x^i y^j \quad \begin{pmatrix} s + m = n \\ g_{ij} - \mathrm{const} \end{pmatrix}.$$

The coefficients g_{ij} are expressed in terms of p_{ij} according to the scheme elucidated in /2/, say.

We will consider the case of uniform pressure on the crack edges $p(x, y) = p = \mathrm{const}$. Solving the system of Eqs.(3.2) successively as was done in /2/, say, we obtain

$$\gamma(x, y) = \frac{1}{T(\lambda)}\,\frac{bp}{2GE(k)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2}[1 - \varepsilon A(\lambda)B(k)] + O(\varepsilon^2)$$

$$A(\lambda) = \frac{M(\lambda)}{2T(\lambda)}, \quad B(k) = \frac{(2 - k^2)E(k) - 2(1 - k^2)K(k)}{k^2E(k)}; \; k = \left(1 - \frac{b^2}{a^2}\right)^{1/2}.$$

To be specific, we assume here that $a \geqslant b$; $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively.

Therefore, the result (3.3) enables us to conclude that in the case when the initial stretch coefficients are sufficiently close in both directions, as when they are equal, the prestrain will not change the order of the singularity in the neighbourhood of the crack contour. The normal stress intensity factor is here proportional to the corresponding quantity in the classical problem $(\lambda = 1,\ \varepsilon = 0)$.

It is convenient to consider $N_{\mathrm{I}} = K_{\mathrm{I}0}/K_{\mathrm{I}*}$ as the parameter characterizing the change in the normal stress intensity factor because of prestrain, where $K_{\mathrm{I}0}$ is the normal stress intensity factor in the neighbourhood of an elliptical crack contour when the body is subjected to finite prestrain. The stretch intensity is identical in both directions $(\lambda_1 = \lambda_2 = \lambda)$. The quantity $K_{\mathrm{I}*}$ corresponds to the case of no prestrain $(\lambda = 1)$.

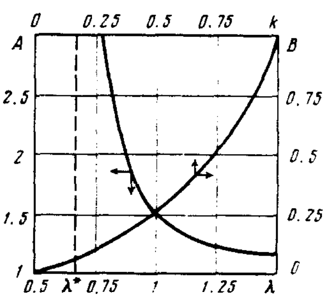

Fig.2

A graph of the change in the parameter $N_{\mathrm{I}} = 1/T(\lambda)$ is in Fig.1. If the body is subjected to initial stretch $(\lambda > 1)$, then $N_{\mathrm{I}} < 1$. This indicates that the initial biaxial tension in the plane of the crack will contribute to strengthening of the body compared with the classical case $\lambda = 1$. In other words, initial biaxial finite stretch reduces the value of the normal stress intensity factor. In the case when the body is subjected to biaxial compression in the plane of the crack $\lambda^* < \lambda < 1$, the value of the normal stress intensity factor increases. This case corresponds to a reduction in the carrying capacity of the body containing the plane crack.

In the case when the prestrain coefficients λ_1 and λ_2 differ slightly, the normal stress intensity coefficient $K_{\mathrm{I}\varepsilon}$ should be calculated from the formula $K_{\mathrm{I}\varepsilon} = N_{\mathrm{I}}[1 - \varepsilon B(k)A(\lambda)]K_{\mathrm{I}*} + O(\varepsilon^2)$.

Changes in the coefficients $A(\lambda)$ and $B(k)$ are shown in Fig.2 for the appropriate parameters.

We note that although the results are obtained for a neo-Hookean material, the proposed method of investigating spatial problems concerning cracks in prestressed geometrically non-linear elastic media is applicable for any hyperelastic incompressible material. The fundamental qualitative deductions also hold here.

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